Glacial erosion model

The ice thickness, \( h \), is computed by solving the equation of mass conservation (Cuffey and Paterson 2010):

\[
\frac{\partial h}{\partial t} = M - \nabla \cdot \mathbf{q} \quad (1)
\]

where \( \mathbf{q} \) is the vertically averaged ice flux (\( \mathbf{q} = h \mathbf{u} \), where \( \mathbf{u} \) is the vertically integrated horizontal ice velocity) and \( M \) the surface mass balance. The ice velocity, \( \mathbf{u} \), is the sum of the deformation and the sliding velocities. The shallow ice approximation is used (Mahaffy 1976, Hutter 1983) to compute Equation 1 with efficiency,

\[
\frac{\partial h}{\partial t} = M - \nabla \cdot \left[ \left( \frac{2B}{\eta + 2} (\rho g)^{n} h^{n+2} + B_s (\rho g)^{n} h^{n} \right) |\nabla (h + z)|^{n-1} \nabla (h + z) \right] \quad (2)
\]

where \( B \) is the ice flow-law parameter \( (6.8 \times 10^{-24} \text{ Pa}^{-3} \text{s}^{-1}) \), \( B_s \) the sliding law parameter \( (3.4 \times 10^{-18} \text{ Pa}^{-3} \text{s}^{-1}) \), \( \rho \) the density of the ice \( (910 \text{ kg/m}) \), \( g \) the gravitational acceleration \( (9.81 \text{ m/s}) \), \( n \) Glen’s Flow parameter \( (3) \), and \( z \) the bedrock topography. Equation 2 is solved using the finite difference method. The model is run until steady state is reached. We use the GEBCO One Minute Grid digital elevation model for the topography, which was re-interpolated at a 5 km resolution to make sure the shallow ice approximation remains valid. The sliding velocity, \( \mathbf{u}_s \), is then calculated

\[
\mathbf{u}_s = B_s (\rho g)^{n} h^{n-1} |\nabla (h + z)|^{n-1} \nabla (h + z) \quad (3)
\]

where the temperature of the ice at the ice-bedrock interface is at melting point (see below for the treatment of the temperature). Finally, erosion rates are computed as follows

\[
\dot{e} = K_g |\mathbf{u}_s|^l \quad (4)
\]

where \( \dot{e} \) is the erosion rate, \( K_g \) is an erosion constant \( (5 \times 10^{-5}) \) and \( l \) the erosion law exponent set equal to 1 (Humphrey and Raymond 1994).
Finally, we reduce the complexity of the model to concentrate on the effects of precipitation and temperature. The model does not include calving or a grounding line, which may modify the model extent, particularly for the western regions. Our goal is solely to highlight the impact of a shift of southern Westerly winds and its effect on precipitation using a simple model. We show that higher precipitation rates about 44°S are required to explain the ice extent. It implies higher flux and higher erosion rates.

**Ice surface mass balance calculation**

The surface mass balance model is specified using an empirical parameterization (Oerlemans 1997, Giesen and Oerlemans 2012), with mass gain resulting from snow precipitation, $P_s$, and mass loss due to the surface energy balance, $Q$, and is computed as follows

$$M = \int P_s + \min(0, \frac{-Q}{\rho_w L_f}) \, dt$$  \hspace{1cm} (5)

where $\rho_w$ is the density of water (1000 kg/m$^3$) and $L_f$ (3.34x10$^5$ J/Kg) is the latent heat of fusion. $Q$ is computed as follows

$$Q = \max(-25, -25 + 20 T_s)$$  \hspace{1cm} (6)

where $T_s$ is the surface temperature (see treatment of temperature below). Total precipitation accounts for changes in altitude and Westerly wind direction through a heuristic parameterization of orographic precipitation,

$$P = \left[ \frac{C_0 - C_1 \int_0^x P(x') \, dx'}{\phi} \right] z$$  \hspace{1cm} (7)

where

$$C_0 = a e^{-\frac{y - y_p}{\phi}} + 3 \times 10^5$$  \hspace{1cm} (8)

in which $y$ is the distance along the latitudinal direction, $C_1$ is a constant (0.25x10$^8$ m$^2$) and $\phi$ sets the width of peak precipitation (150 km). $y_p$ is chosen such that peak precipitation is centered about 44°S in the first two experiments, and about 50°S in the third one. $a$ is an *ad hoc*, adjustable parameter that
controls the magnitude of precipitation. It is set to $9 \times 10^6 \text{ m}^2/\text{yr}$ in the first experiment, $3 \times 10^6 \text{ m}^2/\text{yr}$ in the second and $2.5 \times 10^6 \text{ m}^2/\text{yr}$ in the third one. The difference in precipitation between the first and third simulation is slightly higher than the 2-fold increase suggested in the literature (Moreno et al. 1999, Rojas et al. 2009). However, $\alpha$ was adjusted to produce comparable max accumulation rates for both simulations. Keeping high precipitation rates by setting $\alpha$ to $4.5 \times 10^6 \text{ m}^2/\text{yr}$ in the third simulation leads to accumulation rates higher than $10 \text{ m/yr}$ because of the energy term in the mass balance model (Equation 5). It produces an even larger ice sheet in the southern parts of the model, but it leads to the same conclusions.

Precipitation only falls as snow ($P_s$) where $T_s$ is less than $2^\circ\text{C}$ (Auer 1974). We assume wind comes from the west, although in reality it was more likely coming in a northwest direction. Ultimately, $M$ is filtered using a local regression method and then shifted 30 km to the west to resemble mass balance typically observed on large ice sheets and to better fit the ice extent. The resulting ice mass balance is shown in Figure DR1. We also show in Figure DR2, the distributions of ice thickness, surface temperature and mass balance estimate for Experiment 1 (Figure 2a and Figure DR1a).

**Surface and basal temperature**

The ice surface and basal temperatures are required to compute the mass balance and estimate where the ice is sliding, or not sliding. Annual surface temperature accounts for altitudinal and latitudinal variations and is computed as follows

$$T_s = T_0 - \gamma_1 y - \gamma_2 (z + h)$$ (9)

where $\gamma_1$ is the latitudinal temperature gradient ($5.9 \times 10^{-6} \text{ ^\circ C/m}$), $\gamma_2$ the altitudinal lapse rate ($5 \times 10^{-3} \text{ ^\circ C/m}$) and $T_0$ the temperature at sea level in the top left corner of the model domain (see calculation below). The basal temperature is computed using a 1D analytical solution of the heat transfer equation (Robin 1955, Clarke et al. 1977). In the accumulation area, the basal temperature is given by
\[ T_b = T_s + \frac{q}{k} \frac{hM}{2M} \text{erf} \left( \frac{hM}{2\kappa} \right) \]

(10)

and in the ablation area by

\[ T_b = T_s + h \frac{q}{k} \int_0^1 e^{\frac{hM}{2\kappa}(1-\xi)^2} d\xi \]

(11)

where \( q \) is the surface heat flow (70 mW.m\(^{-2}\)), \( k \) is the thermal conductivity of the underlying bedrock (2.35 W/(mK)) and \( \kappa \) is the thermal diffusivity of ice (1.22x10\(^{-6}\) m\(^2\)/s). Note, this is not an accurate model for ice temperature because the ice cannot exceed the melting point and horizontal conduction and advection are ignored. Sliding is computed where \( T_b \) is greater than the melting temperature \( T_m \).

\[ T_m = -8.7 \times 10^{-4} h \]

(12)

Finally, we use the Precis-DFG climate data to calculate \( T_0 \) at the LGM. This is a 0.25° gridded data set including precipitation, surface temperature, surface humidity, and wind speed and direction for the period 1960-1990. It currently shows a mean annual temperature of about 15 °C at 40°S. Assuming that the mean annual temperature was about 5 to 6°C lower during glacial maxima (e.g. Porter 1981, Clapperton 1994, Hulton et al. 1994), we set \( T_0 \) equal to 9.4°C at 40°S. The resulting basal conditions (i.e. frozen- vs. warm-based) are shown in Figure DR3.

To further assess the whether the ice was frozen to its bed in the southern Patagonian Andes, we ran three additional simulations in which vary \( T_0 \), keeping all other parameters constant to Experiment 1. The results are shown in Figure DR4. It shows it was unlikely that the ice was frozen in valleys during glacial maxima.

**Data compilation**

The compilation of thermochronometric data includes published apatite (U-Th)/He, apatite fission track, zircon (U-Th)/He and zircon fission track ages. We modified data compared to the available global database (Herman et al. 2013),
including corrections for mislocated samples and screened for samples that may have been affected by recent volcanic activity. Finally, we included Al-in hornblende geobarometry data (Hervé 1995, Seifert et al. 2005, Leuthold et al. 2012). In that case, we only used the depth of emplacement and associated age in the inversion approach described below. The data are shown in Figure DR5.

**Inversion of thermochronometric data**

We use a modified version of the method recently developed by Fox et al. (2014), and used in Herman et al. (2013), to invert thermochronometric datasets. This method exploits the information contained in both age-elevation profiles and multi-thermochronometric systems strategies. In this approach, the depth to the closure temperature is expressed as the integral of erosion rate from the thermochronometric age to present-day,

\[ z_c = \int_0^\tau \dot{e} \, dt \]  

where \( z_c \) is the closure depth, \( \tau \) is the thermochronometric age and \( \dot{e} \) the erosion rate. To impose a positivity constraint on our inverse problem, we perform a change of variable in the logarithmic space where

\[ \zeta = \ln (z_c) \]  

\[ \varepsilon = \ln (\dot{e}) \]  

which we include in Equation 13 that becomes

\[ \zeta = \ln (\int_0^\tau \exp(\varepsilon) \, dt) \]  

This forms the inverse problem we wish to solve for \( \varepsilon \). This can be achieved given that \( \zeta \) is estimated. To compute \( \zeta \), we first compute \( z_c \) using the same method as described in detail in Fox et al. (2014), and then take the logarithm of the solution. This inverse problem is weakly non-linear and can be solved using the least-squares method (e.g. Tarantola 2005). We achieve this by first discretizing the integral in Equation 16. It then becomes a summation in which the erosion rate (i.e. exponential of \( \varepsilon \)) is parameterized as a piecewise constant function over fixed time intervals. Similarly to Fox et al. (2014), we impose the
condition that \( \varepsilon \) is correlated in space by defining an a priori model covariance matrix, \( C_M \). This matrix is constructed for all time intervals using the horizontal distance between the \( i \)th and \( j \)th data points, \( d \), and a Gaussian correlation function,

\[
C_M(i,j) = \sigma^2 \exp \left( -\left( \frac{d}{L} \right)^2 \right) \tag{17}
\]

where \( L \) is a specified correlation length, \( d \) the distance between samples, and \( \sigma^2 \) is the a priori variance, which serves primarily as a weighing factor. It is worth noting that this covariance matrix simply implies that samples close to each other must follow the same erosion history and that samples far apart follow independent histories. Finally, both the temperature field and closure depth calculations depend on the solution (i.e. estimated erosion rates), which implies a second non-linearity (Fox et al. 2014).

The non-linear problem is solved using the steepest descent algorithm (Tarantola 2005, p.70),

\[
\varepsilon_{m+1} = \varepsilon_m + \mu \left( C_M G^T C_D^{-1} (\zeta_m - \zeta_{obs}) + (\varepsilon_m - \varepsilon_{pr}) \right) \tag{18}
\]

where \( m \) is the number of iterations, \( C_D \) is the data covariance matrix (which is a diagonal matrix) and \( \mu \) is an ad hoc parameter chosen by trial and error. The model and data covariance are chosen to minimize tradeoff between model and data variance, \( \sigma_d^2 \), (e.g. Aster et al. 2012) \( (\sigma_d = 0.6, L=30 \text{ km and } \sigma = 1.8) \). We start the iterative process using the a priori expected value of \( \varepsilon_{pr} \) \( (\varepsilon_{pr} = \ln (0.4)) \) and \( G \) corresponds to:

\[
G(i,j) = \frac{\partial \ln (\sum \exp(\varepsilon) \Delta t)}{\partial \varepsilon_j} \tag{19}
\]

which can be computed analytically,

\[
G(i,j) = \frac{\exp(\varepsilon_j) \Delta t}{\varepsilon_{c,j}} \tag{20}
\]

Ultimately, the erosion rates, \( \dot{\varepsilon} \), are computed by taking the exponential of \( \varepsilon \).
The misfit function, $S$, that we minimize during the iterative process (Figure DR6) corresponds to

$$2S(\varepsilon) = (\zeta_m - \zeta_{obs})^t C_D^{-1} (\zeta_m - \zeta_{obs}) + (\varepsilon_m - \varepsilon_{pr})^t C_M^{-1} (\varepsilon_m - \varepsilon_{pr})$$

Finally, the posterior covariance, $\tilde{C}$, corresponds to

$$\tilde{C} = C_M - C_M G^t (G C_M G^t + C_D)^{-1} G C_M$$

where the diagonal elements of $\tilde{C}$ give the a posteriori variance, $\sigma_{pa}^2$ (Figure DR7). The a posteriori variance is a measure of the uncertainty on the parameter estimate with respect to the a priori variance. The ratio between a priori and a posteriori variance indicates whether the inclusion of data allows gaining information on the estimated exhumation rate.
Figure DR1. Prescribed mass balance. (a) Prescribed surface mass balance from Experiment 1, (b) for Experiment 2 and (c) for Experiment 3. The blue zone in (a) highlights the effect of increased precipitation on the surface mass balance. Red contours highlight the region where erosion rates are higher than 0.6 mm/yr (i.e. erosion hotspot) in the inversion results shown in Figure 4.
Figure DR2. Distribution of ice thickness, surface temperature and mass balance for Experiment 1. Relative frequency (blue bars) and cumulative distribution (red line) for (a) surface temperature, (b) accumulation rate and (c) ice thickness in the accumulation area for the ice model shown in Figure 2a (Experiment 1).
Figure DR3. Basal temperature conditions. The panels show where the ice is frozen- or warm-based. The three panels correspond to the numerical experiments shown in Figure 2.
Figure DR4. Basal temperature conditions for different surface temperature conditions. Each model corresponds to a different prescribed value for $T_0$. 
Figure DR5. Thermochronometric and geobarometric data.
Figure DR6. Inversion residuals. The standard error on the residuals is nearly twice smaller than the data standard error. $2S$ corresponds to $\chi^2$, and the degree of freedom is set by the amount of data used in the inversion (Tarantola 2005, p.74), i.e. 495. (To avoid computing the inverse of the covariance matrix, we only include the diagonal terms, which lead to a slight underestimation of the $\chi^2$. The actual reduced $\chi^2$ is close to 1.)
Figure DR7. Variance reduction. The panels show the variance reduction, i.e. ratio between a posteriori and a priori standard deviations, $\sigma/\sigma_{po}$, for the last 6 Myr.
Data Repository References


Tarantola, A. Inverse Problem Theory (SIAM, 2005).