APPENDIX
Tidally induced groundwater circulation in an unconfined coastal aquifer modeled with a Hele-Shaw cell

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SCALING OF THE CONTINUITY AND MOMENTUM EQUATIONS
The continuity and momentum equations – (1) and (2) in the text – can be readily scaled within the context of our Hele-Shaw experiments, such that (2) reduces to a simple balance of viscous, gravitational and pressure forces parallel to $x$ and $z$, as embodied in (3) and (4). This follows a straightforward procedure, of which various versions are presented, for example, in Bear (1972), Acheson (1990), Furbish (1997) and Mango (2001).

We start by writing (1) and (2) in component form:

\[
\begin{align*}
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} &= 0, \quad (A1) \\
\frac{u}{\partial x} + \frac{\partial u}{\partial y} + w \frac{\partial u}{\partial z} + \frac{\partial u}{\partial t} &= - \frac{1}{\rho} \frac{\partial p}{\partial x} + \frac{\mu}{\rho} \left( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right), \quad (A2a) \\
\frac{u}{\partial x} + \frac{\partial v}{\partial y} + w \frac{\partial v}{\partial z} + \frac{\partial v}{\partial t} &= - \frac{1}{\rho} \frac{\partial p}{\partial y} + \frac{\mu}{\rho} \left( \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2} \right) \quad \text{and} \quad (A2b) \\
\frac{u}{\partial x} + \frac{\partial w}{\partial y} + w \frac{\partial w}{\partial z} + \frac{\partial w}{\partial t} &= - \frac{1}{\rho} \frac{\partial p}{\partial z} - g + \frac{\mu}{\rho} \left( \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2} \right). \quad (A2c)
\end{align*}
\]

We now define the following dimensionless quantities denoted by circumflexes:

\[
\begin{align*}
\hat{u} &= \hat{U} \hat{u}, \quad \hat{v} = \hat{V} \hat{v}, \quad \hat{w} = \hat{U} \hat{w}, \\
x &= \lambda \hat{x}, \quad y = b \hat{y}, \quad z = \lambda \hat{z} \quad \text{and} \quad t = \tau \hat{t}.
\end{align*}
\]

Here, $\hat{U}$ and $\hat{V}$ are characteristic velocities components, $\lambda$ is a wave attenuation length scale, $b$ is the aperture half-width and $\tau$ is the tidal period. For varying flow driven by a periodic boundary condition (as in tidally induced groundwater flow), the length scale $\lambda$ is set by the propagation distance of waveforms induced by this boundary condition. In a confined aquifer, $\lambda$ is associated

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with a pressure waveform. In the system described herein (analogous to an unconfined aquifer), \( \lambda \) is defined by the water-table waveform (Mango, 2001; Mango et al., 2001); it is determined by the period \( \tau \) of the boundary condition and the cell permeability, and may weakly depend on the amplitude (tidal range) of the boundary condition.

By scaling both \( u \) and \( w \) by \( U \) in (A3), we are leaving open the possibility that these velocity components are of similar magnitude. The magnitude of \( V \) (and therefore of \( v \)) is to be inferred below. By scaling \( x \) and \( z \) by \( \lambda \), we are assuming that the horizontal and vertical distances over which fluid motion occurs are of the same order. (We note that one may choose another length scale for \( z \), for example, the saturated thickness \( B \); but it is straightforward to demonstrate that such a choice would not change the conclusions obtained below.) The aperture half-width \( b \) is the natural choice for scaling \( y \). The period \( \tau \) is the natural choice for scaling \( t \), as this is the time scale over which velocity fluctuations occur.

Substituting the relevant expressions in (A3) into (A1) then obtains:

\[
\frac{U}{\lambda} \frac{\partial u}{\partial x} + \frac{V}{b} \frac{\partial v}{\partial y} + \frac{U}{\lambda} \frac{\partial w}{\partial z} = 0. \tag{A4}
\]

Comparing the leading coefficients in (A4), the first and third terms are of similar magnitude, and it follows that:

\[
V \sim \frac{bU}{\lambda}. \tag{A5}
\]

Thus, the horizontal velocity component \( v \) must be much smaller than either \( u \) or \( w \), so long as \( b < \lambda \). (One may alternatively assert, based on the flow geometry in the Hele-Shaw cell, that \( v \) is everywhere zero, and thereby immediately neglect the second term in (A1), and eliminate (A2b) from the analysis. This would require assuming, a priori, that inertial forces are small relative to viscous forces. So for completeness, we retain (A2b) here.)

Turning to the momentum equations, (A2a-c), we first note that, on physical grounds, the pressure terms in all three equations, and the gravitational term in (A2c), must enter the momentum balance at lowest order in each of the three cases, as these terms provide the basic forces producing fluid motion. For this reason it is unnecessary to scale these terms. Substituting the expressions in (A3) and (A5) into (A2a-c) then obtains:

\[
\frac{U^2}{\lambda} \frac{\partial u}{\partial x} + \frac{U^2}{\lambda} \frac{\partial v}{\partial y} + \frac{U^2}{\lambda} \frac{\partial w}{\partial z} + \frac{U}{\tau} \frac{\partial u}{\partial t} = -\frac{1}{\rho} \frac{\partial p}{\partial x} + \frac{\mu}{\rho} \left( \frac{U}{\lambda^2} \frac{\partial^2 u}{\partial x^2} + \frac{U}{b^2} \frac{\partial^2 v}{\partial y^2} + \frac{U}{\lambda^2} \frac{\partial^2 w}{\partial z^2} \right), \tag{A6a}
\]

\[
\frac{U^2 b}{\lambda^2} \frac{\partial u}{\partial x} + \frac{U^2 b}{\lambda^2} \frac{\partial v}{\partial y} + \frac{U^2 b}{\lambda^2} \frac{\partial w}{\partial z} + \frac{Ub}{\lambda \tau} \frac{\partial \tilde{y}}{\partial t} = -\frac{1}{\rho} \frac{\partial p}{\partial y} + \frac{\mu}{\rho} \left( \frac{Ub}{\lambda^3} \frac{\partial^2 \tilde{y}}{\partial x^2} + \frac{U}{\lambda b} \frac{\partial^2 \tilde{y}}{\partial y^2} + \frac{Ub}{\lambda^3} \frac{\partial^2 \tilde{y}}{\partial z^2} \right) \quad \text{and} \tag{A6b}
\]
Comparing the leading coefficients in (A6a-c), several conclusions immediately obtain. First, with \( b \ll \lambda \), viscous terms involving \( \frac{\partial^2}{\partial y^2} \) are much larger than the other two viscous terms in each equation. Second, the largest viscous term in (A6b) is much smaller than the largest viscous terms in (A6a) and (A6c). Third, with \( \mu/\rho \geq 1 \), the convective terms in each equation are much smaller than the largest corresponding viscous term. Fourth, the inertial terms in (A6b), including the unsteady term, are much smaller than corresponding terms in (A6a) and (A6c). Further, the pressure term in (A6b), which can be no larger than the viscous term in (A6b), is much smaller than the pressure terms in (A6a) and (A6c). It follows from these points that (A6b) can be neglected in the remainder of the analysis, and that (A6a) and (A6c) reduce to:

\[
\begin{align*}
\frac{U}{\tau} \frac{\partial \hat{u}}{\partial t} &= -\frac{1}{\rho} \frac{\partial p}{\partial x} + \frac{\mu}{\rho} \frac{U}{b^2} \frac{\partial^2 \hat{u}}{\partial y^2} \\
\frac{U}{\tau} \frac{\partial \hat{w}}{\partial t} &= -\frac{1}{\rho} \frac{\partial p}{\partial z} - g + \frac{\mu}{\rho} \frac{U}{b^2} \frac{\partial^2 \hat{w}}{\partial y^2}.
\end{align*}
\]  

(A7a)

(A7b)

It now remains to show that the unsteady terms in (A7a-b) can be neglected relative to other terms. Recall that the pressure and gravitational terms must enter the momentum balance at lowest order, which means that at least one additional term must be retained in each of (A7a) and (A7b). It immediately follows that we can compare the leading coefficients in the unsteady and viscous terms, and conclude that the unsteady terms can be neglected if

\[
\frac{\rho b^2}{\mu} \ll \tau,
\]

whence (A7a) and (A7b) reduce to (3) and (4) in the text, namely:

\[
\begin{align*}
\frac{\partial^2 \tilde{u}}{\partial y^2} &= \frac{1}{\mu} \frac{\partial p}{\partial x} && \text{and} \\
\frac{\partial^2 \tilde{w}}{\partial y^2} &= \frac{\rho g}{\mu} + \frac{1}{\mu} \frac{\partial p}{\partial z}.
\end{align*}
\]  

(3)

(4)

The condition (A8) is readily satisfied in our Hele-Shaw experiments. For example, with \( \rho \approx 1.26 \) g·cm\(^{-3}\), \( b = 0.15 \) cm and \( \mu \approx 9 \) g cm\(^{-1}\)·s\(^{-1}\) (Table 1), then \( \rho b^2/\mu \approx 0.003 \) s, which is much less than the experimental periods \( \tau = 120 \) s (run 1) and \( \tau = 20 \) s (run 2).

We note that the omission of unsteady terms in the momentum balance does not imply that fluid velocities do not change with time. Rather, this result means that (inertial) accelerations are sufficiently small that the local velocity responds immediately to changes in the local pressure gradient. We also note that (4) reveals why the pressure field in general is not hydrostatic. Namely,
only in the absence of the viscous term would (4) reduce to the hydrostatic equation, \( \frac{dp}{dz} = -\rho g \).

**INTEGRATION OF THE REDUCED MOMENTUM EQUATIONS**

To obtain the width-averaged velocity components \( \bar{u} \) and \( \bar{v} \), analogous to Darcy fluxes for groundwater flow, the reduced equations (3) and (4) are first integrated over the cell aperture as follows. Because pressure \( p \) is a function of \( x \) and \( z \), but not \( y \), (3) and (4) can immediately be integrated with respect to \( y \) to obtain:

\[
\frac{\partial u}{\partial y} = \frac{1}{\mu} \frac{\partial p}{\partial x} y + C_1 \quad \text{and} \quad \frac{\partial w}{\partial y} = \frac{1}{\mu} \left( \rho g + \frac{\partial p}{\partial z} \right) y + C_2.
\]  

(A9)

(A10)

The constants of integration, \( C_1 \) and \( C_2 \), are obtained from the symmetry of the flow over the cell aperture, namely that \( \frac{\partial u}{\partial y} = \frac{\partial w}{\partial y} = 0 \) at \( y = 0 \) (the centerline of the cell; Figure 1). Thus, \( C_1 = C_2 = 0 \), so (A9) and (A10) become:

\[
\frac{\partial u}{\partial y} = \frac{1}{\mu} \frac{\partial p}{\partial x} y \quad \text{and} \quad \frac{\partial w}{\partial y} = \frac{1}{\mu} \left( \rho g + \frac{\partial p}{\partial z} \right) y.
\]  

(A11)

(A12)

These are integrated with respect to \( y \) a second time to obtain:

\[
u = \frac{1}{2\mu} \frac{\partial p}{\partial x} y^2 + C_3 \quad \text{and} \quad w = \frac{1}{2\mu} \left( \rho g + \frac{\partial p}{\partial z} \right) y^2 + C_4.
\]  

(A13)

(A14)

The constants of integration, \( C_3 \) and \( C_4 \), are obtained from the no-slip condition, namely that \( u = w = 0 \) at \( y = \pm b \) (the cell walls; Figure 1). Thus, \( C_3 = -(b^2/2\mu)(\partial p/\partial x) \) and \( C_4 = -(b^2/2\mu)(\rho g + \partial p/\partial z) \), so (A13) and (A14) become:

\[
u = -\frac{1}{2\mu} \frac{\partial p}{\partial x} (b^2 - y^2) \quad \text{and} \quad w = -\frac{1}{2\mu} \left( \rho g + \frac{\partial p}{\partial z} \right) (b^2 - y^2).
\]  

(A15)

(A16)

To obtain the width-averaged (mean) velocity components, \( \bar{u} \) and \( \bar{v} \), we apply the mean-value theorem to (A15) and (A16). Namely,

\[
\bar{u} = \frac{1}{2b} \int_{-b}^{b} u(y) dy = -\frac{1}{4\mu b} \frac{\partial p}{\partial x} \int_{-b}^{b} (b^2 - y^2) dy \quad \text{and} \quad \bar{w} = \frac{1}{2b} \int_{-b}^{b} w(y) dy = -\frac{1}{4\mu b} \left( \rho g + \frac{\partial p}{\partial z} \right) \int_{-b}^{b} (b^2 - y^2) dy.
\]  

(A17)

(A18)
Evaluation of the integrals in (A17) and (A18) then gives (5) in the text:

$$\bar{u} = -\frac{b^2}{3\mu} \frac{\partial p}{\partial x}, \quad \bar{w} = -\frac{b^2}{3\mu} \left( \rho g + \frac{\partial p}{\partial z} \right). \tag{5}$$

Here, $b^{2/3}$ is analogous to the permeability. We note that by specifying the total aperture width as $b_0$, then the half-width $b = b_0/2$ and the permeability is $b_0^{2/12}$, which is the quantity preferred by some authors (e.g., Bear, 1972; Domenico and Schwartz, 1990). For geological porous media the permeability is often written as $Cb_0^{2}$, where now $b_0$ represents a characteristic pore diameter or particle diameter. The factor $C$ is a function of pore system geometry, and is determined experimentally or is based on an idealized model of pore geometry (e.g., Bear, 1972).

**EXPERIMENTAL CONDITIONS AND PROCEDURES**

The reservoir:gap volume and surface-area ratios of the apparatus (Fig. 2) are each about 100:1. To within 0.01, the forced “tide” level is therefore independent of fluid exchange with the gap, analogous to the natural setting. The tidal amplitude and period can be readily set by adjusting the length of the arm connections and motor speed (Fig. 2).

The density of the glycerin is insensitive to small (laboratory) temperature changes. Its viscosity, however, may vary with such changes. We therefore monitor viscosity throughout each experimental run by settling (with replication) small high-precision-machined stainless steel spheres in the glycerin (far from boundaries). We calculate the viscosity using Stokes law with settling speeds obtained from stopwatch-timed motion over a distance of 15 cm. Glycerin is also sensitive to humidity. We therefore cover the tank with thin contact-sealing plastic after filling it to minimize adsorption of moisture; and we continuously run dehumidifiers in the lab. Experience indicates that the glycerin viscosity may change only a few percent at most with these precautions (Mango, 2001).

Surface tension associated with the acrylic-glycerin-air combination provides a surface that is very weakly “wetting” – that is, without significant meniscus curvature. The behavior of this surface therefore faithfully mimics a “water table” response to tidal forcing, rather than being controlled by surface-tension forces.

Particle imaging velocimetry (PIV) works by computing the peak in the cross-correlation matrix of 64 × 64 pixel interrogation regions for successive images, typically taken at 5-8 frames per second depending on overall flow speeds. The velocity field is mapped by systematically repositioning the camera over a grid of 6 cm × 6 cm “windows”. Similarly, variations in the local “water table” position can be mapped from time-series images referenced to a grid. We allow tens of tidal cycles to elapse before imaging, to ensure full spin-up to a (dynamic) steady condition, so transient effects related to (static) initial conditions have dissipated. The timing of the images taken at each grid position is referenced to the tidal cycle (Mango, 2001).

The factor 1.4 by which the PIV measurements of velocity overestimate the width-averaged value is obtained as follows. Because the laser sheet illuminates tracer particles within the middle (approximately) one-third of the cell aperture, we apply the mean-value theorem to (A15) and (A16) over this middle one-third, then form the ratio of these results with the averaged values in (5). That is,

$$\frac{1}{u} \frac{3}{2b} \int_{-b/3}^{b/3} u(y) \, dy, \quad \frac{1}{w} \frac{3}{2b} \int_{-b/3}^{b/3} w(y) \, dy. \tag{A19}$$

Upon substituting (A15), (A16) and (5) into (A19) and evaluating the integrals, one obtains the
factor \((3/2)(1 - 1/27) \approx 1.4\) in each case.

**TIME-AVERAGED AND FLUCTUATING PRESSURE FIELDS**

Non-hydrostatic conditions must exist over the length scale of the tidal zone. To show this, we first note that the width-averaged velocity components \(\bar{u}\) and \(\bar{v}\) must satisfy an equation of continuity having the form:

\[
\frac{\partial \bar{u}}{\partial x} + \frac{\partial \bar{w}}{\partial z} = 0. \tag{A20}
\]

Substituting (5) into (A20) and differentiating with respect to \(x\) and \(z\) then obtains:

\[
\frac{\partial^2 p}{\partial x^2} + \frac{\partial^2 p}{\partial z^2} = 0. \tag{A21}
\]

Thus the pressure field satisfies a Laplacian. We now decompose the pressure into a time-averaged part \(P\) and a fluctuation \(p'\) about the time average:

\[
p(x,z,t) = P(x,z) + p'(x,z,t). \tag{A22}
\]

Substituting (A22) into (A21) then obtains:

\[
\frac{\partial^2 P}{\partial x^2} + \frac{\partial^2 P}{\partial z^2} + \frac{\partial^2 p'}{\partial x^2} + \frac{\partial^2 p'}{\partial z^2} = 0. \tag{A23}
\]

By definition the time average of \(p'\) is zero. So upon taking the time average of (A23),

\[
\frac{\partial^2 P}{\partial x^2} + \frac{\partial^2 P}{\partial z^2} = 0. \tag{A24}
\]

Comparing this with (A23), it follows that:

\[
\frac{\partial^2 p'}{\partial x^2} + \frac{\partial^2 p'}{\partial z^2} = 0. \tag{A25}
\]

Thus, (A24) and (A25) indicate that both the time-averaged pressure field and the fluctuating pressure field satisfy a Laplacian. This result is important, because it establishes that the time-averaged subsurface circulation (§3.2) can be associated with a time-averaged pressure field satisfying (A24). This circulation therefore may be viewed as the outcome of a (steady) boundary-value problem that is independent of the fluctuating pressure and velocity fields.

Consider the sloping cell boundary (“sediment-water interface”) seaward of the low-tide level (Fig. 3). By definition the local pressure at this boundary \(p_b = \rho g \psi_b\), where \(\psi_b = \Psi_b(x) + \psi_b'(t)\) is the local fluid depth. Because of the sloping interface, the time-averaged depth \(\Psi_b\) necessarily increases seaward. Let \(z = z_b(x)\) denote the local coordinate of the interface. Then the time-averaged hydraulic head at the interface \(H_b = z_b(x) + \Psi_b(x) = \text{const}\), which characterizes the interface as a “constant head” boundary in the time average. The Dupuit approximation, in contrast, supposes that
vertical velocity components are negligibly small, which is equivalent to assuming that vertical components of the pressure gradient in excess of the hydrostatic gradient are small, or that vertical components of the hydraulic head gradient are small. This in turn would imply that the hydraulic head at the boundary, in the presence of seaward flow associated with the circulation, decreases seaward – which is incompatible with a constant-head condition. Indeed, for isotropic permeability the flow velocity at the interface must be normal to it, requiring a significant vertical component. Near the interface the time-averaged flow field associated with \( P(x, z) \) qualitatively must be reminiscent of (well known) flow conditions near a recharging stream that bounds an unconfined aquifer (e.g., Hubbert, 1940; Bear, 1972). The time-averaged circulation in our experiments (Fig. 3), which involves vertical motions that are of the same order as horizontal motions, therefore clearly reveals non-hydrostatic conditions over the length scale of the tidal zone.

Turning to the fluctuating pressure \( p' \), the source of these fluctuations resides in the changing pressure conditions at the interface boundary and in the fluctuating water-table elevation. The latter, in particular, may be viewed as an “edge” excitation of the pressure field. In this case the Laplacian form of (A25) qualitatively requires that the fluctuations decrease away from the source of the excitation. In physical terms, viscous forces dissipate mechanical energy pumped into the system at its boundaries. The effect of this is that the intensity of oscillating fluid motions generally declines with depth and with distance from the tidal zone.

REFERENCES CITED