Governing Equations

To analyze saline groundwater circulation through carbonate platforms we solve the partial differential equations representing fluid flow, heat and solute transport in a porous medium. Fluid velocities are sufficiently small that the kinetic energy of the water can be neglected, and thus Darcy’s Law is valid. Thus we use Darcy’s law written for a fluid of variable density in Cartesian tensor notation:

$$q_i = -\frac{k_{ij}}{\mu} \left( \frac{\partial P}{\partial x_j} + \rho g \frac{\partial z^*}{\partial x_j} \right)$$

(Equation 1)

where $q_i$ is the specific discharge [LT$^{-1}$], $k_{ij}$ is the intrinsic permeability, a second order tensor [L$^2$], $\mu$ is the dynamic viscosity of the fluid [ML$^{-1}$T$^{-1}$], $P$ is the fluid pressure [ML$^{-1}$T$^{-2}$], $\rho$ is the fluid density [ML$^{-3}$], $g$ is the acceleration due to gravity [LT$^{-2}$], $z^*$ is the elevation of the reference point above a standard datum [L], and $x_j$ is the coordinate direction.

By substituting Darcy’s law into a mass balance equation, one can obtain the groundwater flow equation. In our analysis we assume that fluid pressures in the system change rapidly over time relative to changes in boundary conditions or physical rock properties. Thus a reasonable assumption is that the system is in steady state with respect to fluid pressure. There are no sources of sinks of water other than through the boundaries under constant pressure conditions. The steady state flow equation under these conditions can be written as:

$$\frac{\partial}{\partial x_i} \left[ \frac{\rho k_{ij}}{\mu} \left( \frac{\partial P}{\partial x_j} + \rho g \frac{\partial z^*}{\partial x_j} \right) \right] = 0$$
Sub-surface heat transport occurs by conduction, convection and dispersion. As with fluid mass internal temperatures are assumed to change rapidly relative to changes in boundary conditions or physical rock properties. If we assume that the effect of hydrodynamic dispersion on the heat transport is small relative to conductive heat transport and that there are no internal sources of heat within the platform, the heat transport equation can be written as:

$$\frac{\partial}{\partial x_i} \left( K_i \frac{\partial T}{\partial x_j} \right) - \frac{\partial}{\partial x_i} \left( \rho c_w q_i T \right) + Q_v = 0$$

(equation 3)

where $K$ is the bulk thermal conductivity of the porous medium (EL$^{-1}$t$^{-1}$T$^{-1}$), $T$ is the temperature (T), $c_w$ is the heat capacity of seawater at 25° C [EM$^{-1}$T$^{-1}$], $Q_v$ is the geothermal heat flux [EL$^{-3}$t$^{-1}$], $E$ is the energy [ML$^2$T$^{-2}$] and $t$ is time [t]. We assume that the bulk thermal conductivity of the porous medium is a volumetric average of the thermal conductivities of the water and the rock. The bulk thermal conductivity can be expressed as:

$$K_r = \epsilon K_w + (1-\epsilon)K_s$$

(equation 4)

where $\epsilon$ is the porosity [dimensionless], $K_w$ is the thermal conductivity of the water [EL$^{-1}$t$^{-1}$T$^{-1}$] and $K_s$ is the thermal conductivity of the rock.

The concentration of transported chemical species is controlled by fluid velocities, flow paths, mechanical dispersion, molecular diffusion and geochemical reactions. The transport of mass is treated as a single transient phenomenon. Assuming that salinity is a conservative (non-reactive) solute, the solute transport can be written as:

$$\frac{\partial}{\partial x_i} \rho D \left( \frac{\partial C}{\partial x_i} \right) - \frac{\partial}{\partial x_i} \left( \rho v C \right) + Q_s = \frac{\partial (\rho C)}{\partial t}$$

(equation 5)
where $D$ is the dispersion coefficient [$L^2T^{-1}$], $C$ is the mass concentration per unit volume [$ML^{-3}$] and $Q_s$ is the solute source flux [$ML^{-3}T^{-1}$]. The dispersion coefficient tensor $D$ (Bear, 1972) can be written as:

$$D = \alpha v + D_m$$

(equation 6)

where $\alpha$ is the dispersivity coefficient [$L$], $v$ is the flow velocity [$LT^{-1}$] and $D_m$ is the diffusion coefficient [$L^2T^{-1}$]. For a medium that is isotropic with respect to dispersion it can be shown that the principle components of $\alpha$ occur along the direction of flow and transverse to it (Bear, 1972). In this case the dispersivity tensor reduces to two components: the longitudinal dispersivity $\alpha_L$ and the transverse dispersivity $\alpha_T$.

**Equations of State**

Fluid density is specified to be a non-linear function of temperature (Bolz and Tuve, 1973) and a linear function of salinity (Voss, 1984) by the relation:

$$\rho = (1024.5 - 0.067417 \cdot T - 0.0388 \cdot T^2) + [(C - 35.7) \cdot 0.7]$$

(equation 7)

where fluid density is in kg/m$^3$, temperature is in °C and salinity is in parts per thousand (%o). Fluid viscosity is also allowed to vary with temperature (Meyer et al., 1967) by the relation:

$$\mu = 239.4 \cdot 10^{-7} \cdot 10^{\frac{248.37}{T+133.15}}$$

(equation 8)

where $\mu$ is in units of kg/(m·sec) and $T$ is in °C. Although this relation is valid for pure water, the viscosity of seawater is only about 5% greater than pure water at standard temperatures and pressures. The effect of solute concentration on viscosity is negligible over the range of fluid density investigated in this study (Phillips, 1991; Ophorli, 1998).
Stream functions

One method of illustrating fluid flow is a standard hydrogeological technique using stream functions (Phillips, 1991; Anderson and Woessner, 1992). Stream lines (lines of equal fluid flux derived from contouring stream functions) show both the magnitude and direction of fluid flux. For clarity, a summary of the method we used to generate streamlines is included below. Further details of the method can be found in the above mentioned hydrogeological references.

Stream lines are derived from stream functions. Stream functions (U) are defined as:

\[ qx = - \left( \frac{\partial U}{\partial y} \right), \]

(equation 9)

where: \( qx \) is the flux component in the horizontal direction; and \( \delta y \) is the incremental length in the vertical direction. Stream functions were generated by calculating the \( x \) (horizontal) component of flux for each node in the model grid (derived from the element dimension and output velocity from SUTRA). Stream functions at the no-flow boundaries were set to a constant value of zero. The distribution of stream functions in the platform are calculated by summing, starting at the bottom right hand corner of the grid, the nodal value of stream functions vertically, for each node in the \( y \) (vertical) direction of the grid. This process was repeated for the adjacent column of nodes in turn, sweeping through the entire grid. Contouring the nodal values of stream functions across the grid generates the stream lines (a line of equal fluid flux) that we plot.


Bolz, R.E. and Tuve, G.L., 1973, Handbook of tables for applied engineering science: Cleveland, CRC

v. 204, p. 261-270.


Voss, C., 1984, SUTRA - saturated unsaturated transport - A finite element simulation model for
saturated unsaturated fluid density dependent groundwater flow with energy transport and
chemically-reactive single species transport: U.S. Geological Survey Water Resources Investigative